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Majorization Problems for Certain Subclasses of Meromorphic Multivalent Functions Associated with the Liu-Srivastava Operator

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Abstract. In the present paper, we introduce new subclasses of certain meromorphic multivalent functions defined by a class of linear operators involving the Liu-Srivastava operator, and investigate the majorization properties for functions belonging to these classes. Also, we point out some useful consequences of our main results.

1. Introduction and Preliminaries

Let f and g be two analytic functions in the open unit disk

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}.$$

We say that f is majorized by g in Δ (see [22]) and write

$$f(z) \ll g(z) \quad (z \in \Delta), \tag{1.1}$$

if there exists a function φ, analytic in Δ such that

$$|\varphi(z)| \leq 1 \quad \textit{and} \quad f(z) = \varphi(z)g(z) \quad (z \in \Delta). \tag{1.2}$$

It may be noted here that (1.1) is closely related to the concept of quasi-subordination between analytic functions.

For two functions f and g, analytic in Δ, we say that the function f is subordinate to g in Δ, if there exists a Schwarz function ω, which is analytic in Δ with

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \Delta),$$

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such that

$$f(z) = g(\omega(z)) \quad (z \in \Delta).$$

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function g is univalent in Δ, then

$$f(z) \prec g(z) \quad (z \in \Delta) \Longleftrightarrow f(0) = g(0) \quad \textit{and} \quad f(\Delta) \subset g(\Delta).$$

Let Σ_p denote the class of functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_k z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, \cdots\}),$$

which are analytic and p-valent in the punctured unit disk $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \Delta \setminus \{0\}$. For simplicity, we write $\Sigma = \Sigma_1$.

For functions $f_m \in \Sigma_p$ given by

$$f_m(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k,m} z^{k-p} \quad (m = 1, 2;\ p \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^{k-p} = (f_2 * f_1)(z).$$

For parameters $\alpha_i \in \mathbb{C}$ $(i = 1, 2, \cdots, q)$ and $\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-$ $(\mathbb{Z}_0^- = 0, -1, -2, \cdots;\ j = 1, 2, \cdots, s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ is defined by (see, for example, [24,28])

$${}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_q)_k}{(\beta_1)_k \cdots (\beta_s)_k} \frac{z^k}{k!}$$

$$(q \le s + 1;\ q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\};\ z \in \Delta),$$

where $(\nu)_k$ denotes the Pochhammer symbol defined, in terms of Gamma function, by

$$(\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} = \begin{cases} 1 & (k = 0;\ \nu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}), \\ \nu(\nu + 1) \cdots (\nu + k - 1) & (k \in \mathbb{N};\ \nu \in \mathbb{C}). \end{cases}$$

We now introduce a function $h_p^{\lambda,\mu}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ defined by

$$\begin{aligned} h_p^{\lambda,\mu}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) &= (1 - \lambda + \mu) z^{-p}{}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) \\ +(\lambda - \mu) z [z^{-p}{}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)]' &+ \lambda\mu z^2 [z^{-p}{}_qF_s(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)]'' \quad (1.3) \\ (p \in \mathbb{N};\ \lambda, \mu \ge 0;\ z \in \Delta^*). \end{aligned}$$

In particular, when $\lambda = \mu = 0$, we obtain

$$h_p^{0,0}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) = h_p(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$$

introduced and studied by Liu and Srivastava [19].

Corresponding to the function $h_p^{\lambda,\mu}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z)$ given by (1.3), we consider a linear operator $H_p^{\lambda,\mu}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) : \Sigma_p \to \Sigma_p$ defined by the following Hadamard product (or convolution):

$$H_p^{\lambda,\mu}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s) f(z) = h_p^{\lambda,\mu}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s; z) * f(z). \quad (1.4)$$

For the sake of convenience, we write

$$H_{p,q,s}^{\lambda,\mu}(\alpha_1) = H_p^{\lambda,\mu}(\alpha_1, \cdots, \alpha_q; \beta_1, \cdots, \beta_s).$$

It is easily verified from the definition (1.4) that

$$z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)' = \alpha_1 H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)f(z) - (\alpha_1+p)H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z). \tag{1.5}$$

We note that, for $\lambda = \mu = 0$, the operator $H_{p,q,s}^{0,0}(\alpha_1)$ reduces to the Liu-Srivastava operator $H_{p,q,s}(\alpha_1)$ (see [19,20]; also [5,27]), while the Liu-Srivastava operator is the meromorphic analogous of the Dziok-Srivastava operator (see [6-8]; also [21,25]), which include (as its special cases) the meromorphic analogous of the Carlson-Shaffer linear operator $L_p(a,c) = H_{p,2,1}^{0,0}(1, a; c)$ (see [17,18,33]), the meromorphic analogous of the Ruscheweyh derivative operator $D^{n+1} = L_p(n+p, 1)$ (see [32]), and the operator

$$J_{c,p} = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)dt = L_p(c, c+1) \ \ (c > 0; \ f \in \Sigma_p)$$

studied by Uralegaddi and Somanatha [31].

Using the operator $H_{p,q,s}^{\lambda,\mu}(\alpha_1)$, we now introduce the following subclasses of meromorphic multivalent functions.

Definition 1.1. A function $f \in \Sigma_p$ is said to be in the class $S_{p,q,s}^{\lambda,\mu,m}[\eta; A, B]$ of p-valently meromorphic functions of order η $(0 \le \eta < p)$ in Δ^*, if and only if

$$\frac{1}{p-\eta}\left(-\frac{z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m+1)}}{\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)}} - \eta - m\right) \prec \frac{1+Az}{1+Bz} \tag{1.6}$$

$$(z \in \Delta^*; \ q \le s+1; \ q, s, m \in \mathbb{N}_0; \ p \in \mathbb{N}; \ \lambda, \ \mu \ge 0; \ -1 \le B < A \le 1).$$

Remark 1.1. (i) For $A = 1$ and $B = -1$, we set

$$S_{p,q,s}^{\lambda,\mu,m}[\eta; 1, -1] = S_{p,q,s}^{\lambda,\mu,m}(\eta),$$

where $S_{p,q,s}^{\lambda,\mu,m}(\eta)$ denote the class of functions $f \in \Sigma_p$ satisfying the following inequality:

$$Re\left[\frac{1}{p-\eta}\left(\frac{z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m+1)}}{\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)}} + \eta + m\right)\right] < 0.$$

(ii) Further, for $\lambda = \mu = 0, q = 2, s = 1, \alpha_1 = \beta_1 = 1$ and $\alpha_2 = 1$, we write

$$S_{p,2,1}^{0,0,0}(\eta) = S_p^*(\eta) \ \ and \ \ S_{p,2,1}^{0,0,1}(\eta) = K_p(\eta),$$

which are meromorphic p-valently starlike and meromorphic p-valently convex functions of order η $(0 \le \eta < p)$ in Δ^*, respectively (see Aouf and Xu [4]).

Definition 1.2. A function $f \in \Sigma_p$ is said to be in the class $I_{p,q,s}^{\lambda,\mu,m}[\alpha, b; A, B]$ of p-valently meromorphic spirllike functions of complex order $b \ne 0$ in Δ^*, if and only if

$$\left[1 - \frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m+1)}}{p\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)}} + m + 1\right)\right] \prec \frac{1+Az}{1+Bz} \tag{1.7}$$

$$(z \in \Delta^*; q \le s+1; \ q, s, m \in \mathbb{N}_0; \ p \in \mathbb{N}; \lambda, \ \mu \ge 0; \ b \in \mathbb{C}^*; \ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}; \ -1 \le B < A \le 1).$$

Remark 1.2. (i) For $A = 1$ and $B = -1$, we set

$$I_{p,q,s}^{\lambda,\mu,m}[\alpha, b; 1, -1] = I_{p,q,s}^{\lambda,\mu,m}(\alpha, b),$$

where $I_{p,q,s}^{\lambda,\mu,m}(\alpha, b)$ denote the class of functions $f \in \Sigma_p$ satisfying the following inequality:

$$Re\left[\frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m+1)}}{p\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)}} + m + 1\right)\right] < 1.$$

(ii) Further, for $\lambda = \mu = 0, q = 2, s = 1, \alpha_1 = \alpha_2 = \beta_1 = 1$ and $\alpha = 0$, we write

$$I_{p,2,1}^{0,0,0}(0, b) = S_p^*(b) \ \ and \ \ I_{p,2,1}^{0,0,1}(0, b) = K_p(b),$$

which are meromorphic p-valently starlike and meromorphic p-valently convex functions of complex order b ($b \in \mathbb{C}^*$) in Δ^*, respectively (when $p = 1$, see Aouf [3]).

A majorization problem for the normalized class of starlike functions has been investigated by MacGregor [22] and Altintas et al.[1,2]. Recently, Goyal et al.[13,14], Goswami et al.[10-12], Li et al.[16], Tang et al.[29,30], and Prajapat and Aouf [26] generalized these results for different analytic function classes defined by using various operators. However, until now, only one article deals with the majorization problem for the class of meromorphic functions (see Goyal and Goswami [15]).

The purpose of this paper is to investigate the problems of majorization of the classes $S_{p,q,s}^{\lambda,\mu,m}[\eta; A, B]$ and $I_{p,q,s}^{\lambda,\mu,m}[\alpha, b; A, B]$ defined by the operator $H_{p,q,s}^{\lambda,\mu}(\alpha_1)$, and give some special cases of our main results.

2. Majorization Problem for the Class $S_{p,q,s}^{\lambda,\mu,m}[\eta; A, B]$

We begin by proving the following result.

Theorem 2.1. Let the function $f \in \Sigma_p$ and suppose that $g \in S_{p,q,s}^{\lambda,\mu,m}[\eta; A, B]$. If

$$\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)} \ll \left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)} \quad (z \in \Delta^*;\ m \in \mathbb{N}_0),$$

then

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1 + 1)f(z)\right)^{(m)}\right| \leq \left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1 + 1)g(z)\right)^{(m)}\right| \ \ for \ \ |z| \leq r_1, \tag{2.1}$$

where $r_1 = r_1(p, \alpha_1, m, \eta, A, B)$ is the smallest positive root of the equation

$$|(p - \eta)(B - A) + (\alpha_1 - 2m)B|r^3 - (|\alpha_1 - 2m| + 2|B|)r^2 - (|(p - \eta)(B - A) + (\alpha_1 - 2m)B| + 2)r$$
$$+|\alpha_1 - 2m| = 0, \tag{2.2}$$
$$(m \in \mathbb{N}_0;\ p \in \mathbb{N};\ 0 \leq \eta < p;\ \alpha_1 \in \mathbb{C}^*;\ \ -1 \leq B < A \leq 1).$$

Proof. Since $g \in S_{p,q,s}^{\lambda,\mu,m}[\eta; A, B]$, we find from (1.6) that

$$\frac{1}{p - \eta}\left(-\frac{z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m+1)}}{\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)}} - \eta - m\right) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{2.3}$$

where $\omega(z) = c_1z + c_2z^2 + \cdots$, $\omega \in P$, P denotes the well-known class of the bounded analytic functions in Δ and satisfies the conditions (see, for details, Goodman [9])

$$\omega(0) = 0 \ \ and \ \ |\omega(z)| \leq |z| \ \ (z \in \Delta). \tag{2.4}$$

From (2.3), we get

$$-\frac{z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m+1)}}{\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)}} = \frac{p+m+[A(p-\eta)+B(\eta-m)]\omega(z)}{1+B\omega(z)}. \tag{2.5}$$

Now, using the following, easily verified from (1.5), identity

$$z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m+1)} = \alpha_1\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)} - (\alpha_1+p+m)\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)} \tag{2.6}$$

in (2.5) and making simple calculations, we get

$$\frac{\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}}{\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)}} = \frac{(\alpha_1-2m)+[(p-\eta)(B-A)+(\alpha_1-2m)B]\omega(z)}{\alpha_1[1+B\omega(z)]},$$

which, in view of (2.4), immediately yields the inequality

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)}\right| \leq \frac{|\alpha_1|(1+|B||z|)}{|\alpha_1-2m|-|(p-\eta)(B-A)+(\alpha_1-2m)B||z|} \times \left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}\right|. \tag{2.7}$$

Next, since $\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)}$ is majorized by $\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)}$ in Δ^*, from (1.2), we have

$$\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)} = \varphi(z)\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)}.$$

Differentiating it with respect to z and multiplying by z, we obtain

$$z\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m+1)} = z\varphi'(z)\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)} + z\varphi(z)\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m+1)}.$$

Using (2.6), in the above equation, we get

$$\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)f(z)\right)^{(m)} = \frac{1}{\alpha_1}z\varphi'(z)\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)} + \varphi(z)\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}. \tag{2.8}$$

Therefore, noting that $\varphi \in P$ satisfies the following inequality (see, e.g., Nehari [23])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}, \tag{2.9}$$

and making use of (2.7) and (2.9) in (2.8), we get

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)f(z)\right)^{(m)}\right| \leq \left(|\varphi(z)| + \frac{|z|(1-|\varphi(z)|^2)(1+|B||z|)}{(1-|z|^2)\left[|\alpha_1-2m|-|(p-\eta)(B-A)+(\alpha_1-2m)B||z|\right]}\right) \times \left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}\right|,$$

which, upon setting

$$|z| = r \quad \textit{and} \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1),$$

leads us to the inequality

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)f(z)\right)^{(m)}\right|$$

$$\leq \frac{\Phi(\rho)}{(1-r^2)(|\alpha_1-2m|-|(p-\eta)(B-A)+(\alpha_1-2m)B|r)}\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}\right|,$$

where

$$\Phi(\rho)=-r(1+|B|r)\rho^2+(1-r^2)(|\alpha_1-2m|-|(p-\eta)(B-A)+(\alpha_1-2m)B|r)\rho+r(1+|B|r)$$

takes its maximum value at $\rho=1$, with $r_1=r_1(p,\alpha_1,m,\eta,A,B)$ given by (2.2). Furthermore, if $0\leq\sigma\leq r_1(p,\alpha_1,m,\eta,A,B)$, then the function $\Psi(\rho)$ defined by

$$\Psi(\rho)=-\sigma(1+|B|\sigma)\rho^2+(1-\sigma^2)(|\alpha_1-2m|-|(p-\eta)(B-A)+(\alpha_1-2m)B|\sigma)\rho+\sigma(1+|B|\sigma) \tag{2.10}$$

is an increasing function on the interval $0\leq\rho\leq 1$, so that

$$\Psi(\rho)\leq\Psi(1)=(1-\sigma^2)(|\alpha_1-2m|-|(p-\eta)(B-A)+(\alpha_1-2m)B|\sigma)$$

$$(0\leq\rho\leq 1;\ 0\leq\sigma\leq r_1(p,\alpha_1,m,\eta,A,B).$$

Hence, upon setting $\rho=1$, in (2.10), we conclude that (2.1) of Theorem 2.1 holds true for $|z|\leq r_1(p,\alpha_1,m,\eta,A,B)$, where $r_1(p,\alpha_1,m,\eta,A,B)$ is given by (2.2). We complete the proof of Theorem 2.1.

Setting $A=1$ and $B=-1$ in Theorem 2.1, equation (2.2) becomes

$$|\alpha_1+2(p-m-\eta)|r^3-(|\alpha_1-2m|+2)r^2-(|\alpha_1+2(p-m-\eta)|+2)r+|\alpha_1-2m|=0. \tag{2.11}$$

We observe that $r=-1$ is one of the roots of this equation, and the other two roots are given by

$$|\alpha_1+2(p-m-\eta)|r^2-(|\alpha_1+2(p-m-\eta)|+|\alpha_1-2m|+2)r+|\alpha_1-2m|=0,$$

so we can easily find the smallest positive root of (2.11). Hence, we have the following result.

Corollary 2.1. Let the function $f\in\Sigma_p$ and suppose that $g\in S_{p,q,s}^{\lambda,\mu,m}(\eta)$. If

$$\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)}\ll\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)}\quad(z\in\Delta^*;\ m\in\mathbb{N}_0),$$

then

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)f(z)\right)^{(m)}\right|\leq\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}\right|\quad for\ \ |z|\leq r_2,$$

where

$$r_2=r_2(p,\alpha_1,m,\eta)=\frac{\xi-\sqrt{\xi^2-4|\alpha_1+2(p-m-\eta)||\alpha_1-2m|}}{2|\alpha_1+2(p-m-\eta)|}, \tag{2.12}$$

with $\xi=|\alpha_1+2(p-m-\eta)|+|\alpha_1-2m|+2$, and $\alpha_1\in\mathbb{C}^*;\ m\in\mathbb{N}_0;\ p\in\mathbb{N};\ 0\leq\eta<p$.

As a special case of Corollary 2.1, when $\lambda=\mu=0$, we obtain the following result for the Liu-Srivastava operator $H_{p,q,s}(\alpha_1)$:

Corollary 2.2. Let the function $f\in\Sigma_p$ and suppose that $g\in S_{p,q,s}^{0,0,m}(\eta)$. If

$$\left(H_{p,q,s}(\alpha_1)f(z)\right)^{(m)}\ll\left(H_{p,q,s}(\alpha_1)g(z)\right)^{(m)}\quad(z\in\Delta^*;\ m\in\mathbb{N}_0),$$

then

$$\left|\left(H_{p,q,s}(\alpha_1+1)f(z)\right)^{(m)}\right|\leq\left|\left(H_{p,q,s}(\alpha_1+1)g(z)\right)^{(m)}\right|\quad for\ \ |z|\leq r_2,$$

where r_2 is given by (2.12).

Next, let us note that

$$H_{1,2,1}(1,1;1)f(z) = f(z) \ \ and \ \ H_{1,2,1}(2,1;1)f(z) = 2f(z) + zf'(z).$$

If we choose $p = 1, q = 2, s = 1, \eta = 0$, $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = 1$, then, for $m = 0$ and $m = 1$, Corollary 2.2 reduces to the following Corollaries 2.3 and 2.4, respectively.

Corollary 2.3. Let the function $f \in \Sigma$ and suppose that $g \in S^{0,0,0}_{1,2,1}(1;0) = S^*$. If $f(z)$ is majorized by $g(z)$ in Δ^*, then

$$\left|2f(z) + zf'(z)\right| \le \left|2g(z) + zg'(z)\right| \quad for \quad |z| \le \frac{3-\sqrt{6}}{3}.$$

Corollary 2.4. Let the function $f \in \Sigma$ and suppose that $g \in S^{0,0,1}_{1,2,1}(1;0) = K$. If $f'(z)$ is majorized by $g'(z)$ in Δ^*, then

$$\left|3f'(z) + zf''(z)\right| \le \left|3g'(z) + zg''(z)\right| \quad for \quad |z| \le 2-\sqrt{3}.$$

3. Majorization Problem for the Class $I^{\lambda,\mu,m}_{p,q,s}[\alpha, b; A, B]$

Next, we state and prove

Theorem 3.1. Let the function $f \in \Sigma_p$ and suppose that $g \in I^{\lambda,\mu,m}_{p,q,s}[\alpha, b; A, B]$. If

$$\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1)f(z)\right)^{(m)} \ll \left(H^{\lambda,\mu}_{p,q,s}(\alpha_1)g(z)\right)^{(m)} \quad (z \in \Delta^*;\ m \in \mathbb{N}_0),$$

then

$$\left|\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1+1)f(z)\right)^{(m)}\right| \le \left|\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1+1)g(z)\right)^{(m)}\right| \ \ for \ \ |z| \le r_1', \tag{3.1}$$

where $r_1' = r_1'(p, \alpha_1, m, \alpha, b, A, B)$ is the smallest positive root of the equation

$$\begin{aligned}&\left|pb\cos\alpha(B-A) + (\alpha_1 + (1-p)m)Be^{i\alpha}\right| r^3 - \left(\left|\alpha_1 + (1-p)m\right| + 2|B|\right)r^2\\ &-\left(\left|pb\cos\alpha(B-A) + (\alpha_1 + (1-p)m)Be^{i\alpha}\right| + 2\right)r + \left|\alpha_1 + (1-p)m\right| = 0,\\ &(m \in \mathbb{N}_0;\ p \in \mathbb{N};\ \alpha_1, b \in \mathbb{C}^*;\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2};\ -1 \le B < A \le 1).\end{aligned} \tag{3.2}$$

Proof. Since $g \in I^{\lambda,\mu,m}_{p,q,s}[\alpha, b; A, B]$, it follows from (1.7) that

$$1 - \frac{e^{i\alpha}}{b\cos\alpha}\left(\frac{z\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1)g(z)\right)^{(m+1)}}{p\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1)g(z)\right)^{(m)}} + m + 1\right) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \tag{3.3}$$

where $\omega(z)$ is defined as (2.4).

From (3.3), we have

$$-\frac{z\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1)g(z)\right)^{(m+1)}}{p\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1)g(z)\right)^{(m)}} = \frac{(m+1)e^{i\alpha} - [b\cos\alpha(B-A) - (m+1)Be^{i\alpha}]\omega(z)}{e^{i\alpha}[1 + B\omega(z)]}. \tag{3.4}$$

Now, using the identity (2.6) in (3.4) and making simple calculations, we obtain

$$\frac{\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1+1)g(z)\right)^{(m)}}{\left(H^{\lambda,\mu}_{p,q,s}(\alpha_1)g(z)\right)^{(m)}} = \frac{[\alpha_1 + (1-p)m]e^{i\alpha} + [pb\cos\alpha(B-A) + (\alpha_1 + (1-p)m)Be^{i\alpha}]\omega(z)}{\alpha_1 e^{i\alpha}[1 + B\omega(z)]},$$

which, in view of (2.4), immediately yields the following inequality

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)}\right|$$

$$\leq \frac{|\alpha_1|(1+|B||z|)}{|\alpha_1+(1-p)m|-|pb\cos\alpha(B-A)+(\alpha_1+(1-p)m)Be^{i\alpha}||z|}\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}\right|. \tag{3.5}$$

Next, making use of (2.9) and (3.5) in (2.8), and just as the proof of Theorem 2.1, we have

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)f(z)\right)^{(m)}\right|$$

$$\leq \left(|\varphi(z)|+\frac{|z|(1-|\varphi(z)|^2)(1+|B||z|)}{(1-|z|^2)[|\alpha_1+(1-p)m|-|pb\cos\alpha(B-A)+(\alpha_1+(1-p)m)Be^{i\alpha}||z|]}\right)$$

$$\times\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}\right|,$$

which, upon setting $|z|=r$ and $|\varphi(z)|=\rho$ $(0\leq\rho\leq 1)$, leads us to the inequality

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)f(z)\right)^{(m)}\right|$$

$$\leq \frac{\Phi_1(\rho)}{(1-r^2)[|\alpha_1+(1-p)m|-|pb\cos\alpha(B-A)+(\alpha_1+(1-p)m)Be^{i\alpha}|r]}$$

$$\times\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}\right|, \tag{3.6}$$

where the function $\Phi_1(\rho)$ defined by

$$\Phi_1(\rho)=-r(1+|B|r)\rho^2+(1-r^2)[|\alpha_1+(1-p)m|-|pb\cos\alpha(B-A)+(\alpha_1+(1-p)m)Be^{i\alpha}|r]\rho$$

$$+r(1+|B|r)$$

takes its maximum value at $\rho=1$ with $r_1'=r_1'(p,\alpha_1,m,\alpha,b,A,B)$ given by (3.2). Moreover, if $0\leq\delta\leq r_1'(p,\alpha_1,m,\alpha,b,A,B)$, then the function

$$\Psi_1(\rho)=-\delta(1+|B|\delta)\rho^2+(1-\delta^2)[|\alpha_1+(1-p)m|-|pb\cos\alpha(B-A)+(\alpha_1+(1-p)m)Be^{i\alpha}|\delta]\rho$$

$$+\delta(1+|B|\delta)$$

increases on the interval $0\leq\rho\leq 1$, so that $\Psi_1(\rho)$ does not exceed

$$\Psi_1(1)=(1-\delta^2)[|\alpha_1+(1-p)m|-|pb\cos\alpha(B-A)+(\alpha_1+(1-p)m)Be^{i\alpha}|\delta]$$

$$(0\leq\delta\leq r_1'(p,\alpha_1,m,\alpha,b,A,B)).$$

Therefore, from this fact, (3.6) gives the inequality (3.1). This completes the proof of Theorem 3.1.

Setting $A=1$ and $B=-1$ in Theorem 3.1, equation (3.2) becomes

$$\left|2pb\cos\alpha+(\alpha_1+(1-p)m)e^{i\alpha}\right|r^3-(\left|\alpha_1+(1-p)m\right|+2)r^2-(|2pb\cos\alpha+(\alpha_1+(1-p)m)e^{i\alpha}|+2)r$$

$$+|\alpha_1+(1-p)m|=0. \tag{3.7}$$

We observe that $r=-1$ is one of the roots of this equation, and the other two roots are given by

$$\left|2pb\cos\alpha+(\alpha_1+(1-p)m)e^{i\alpha}\right|r^2-(\left|2pb\cos\alpha+(\alpha_1+(1-p)m)e^{i\alpha}\right|+|\alpha_1+(1-p)m|+2)r$$

$$+|\alpha_1 + (1-p)m| = 0,$$

so we can easily find the smallest positive root of (3.7). Hence, we have the following result.

Corollary 3.1. Let the function $f \in \Sigma_p$ and suppose that $g \in I_{p,q,s}^{\lambda,\mu,m}(\alpha, b)$. If

$$\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)f(z)\right)^{(m)} \ll \left(H_{p,q,s}^{\lambda,\mu}(\alpha_1)g(z)\right)^{(m)} \quad (z \in \Delta^*;\ m \in \mathbb{N}_0),$$

then

$$\left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)f(z)\right)^{(m)}\right| \le \left|\left(H_{p,q,s}^{\lambda,\mu}(\alpha_1+1)g(z)\right)^{(m)}\right| \quad for \ \ |z| \le r_2',$$

where

$$r_2' = r_2'(p, \alpha_1, m, \alpha, b) = \frac{\zeta - \sqrt{\zeta^2 - 4\left|2pb\cos\alpha + (\alpha_1 + (1-p)m)e^{i\alpha}\right| |\alpha_1 + (1-p)m|}}{2\left|2pb\cos\alpha + (\alpha_1 + (1-p)m)e^{i\alpha}\right|}, \tag{3.8}$$

with $\zeta = \left|2pb\cos\alpha + (\alpha_1 + (1-p)m)e^{i\alpha}\right| + |\alpha_1 + (1-p)m| + 2$, and $m \in \mathbb{N}_0;\ p \in \mathbb{N};\ \alpha_1, b \in \mathbb{C}^*;\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

Putting $\lambda = \mu = 0$ in Corollary 3.1, we obtain the following result for the Liu-Srivastava operator $H_{p,q,s}(\alpha_1)$:

Corollary 3.2. Let the function $f \in \Sigma_p$ and suppose that $g \in I_{p,q,s}^{0,0,m}(\alpha, b)$. If

$$\left(H_{p,q,s}(\alpha_1)f(z)\right)^{(m)} \ll \left(H_{p,q,s}(\alpha_1)g(z)\right)^{(m)} \quad (z \in \Delta^*;\ m \in \mathbb{N}_0),$$

then

$$\left|\left(H_{p,q,s}(\alpha_1+1)f(z)\right)^{(m)}\right| \le \left|\left(H_{p,q,s}(\alpha_1+1)g(z)\right)^{(m)}\right| \quad for \ \ |z| \le r_2',$$

where r_2' is given by (3.8).

Further, putting $p = 1, q = 2, s = 1, m = 0, \alpha_1 = \beta_1 = 1$ and $\alpha_2 = b = 1$ in Corollary 3.2, we get the following result.

Corollary 3.3. Let the function $f \in \Sigma$ and suppose that $g \in I_{1,2,1}^{0,0,0}(\alpha, 1)$. If $f(z)$ is majorized by $g(z)$ in Δ^*, then

$$\left|2f(z) + zf'(z)\right| \le \left|2g(z) + zg'(z)\right| \quad for \ \ |z| \le r_5,$$

where

$$r_5 = r_5(\alpha) = \frac{\varrho - \sqrt{\varrho^2 - 4\left|2\cos\alpha + e^{i\alpha}\right|}}{2\left|2\cos\alpha + e^{i\alpha}\right|} \quad (\varrho = \left|2\cos\alpha + e^{i\alpha}\right| + 3;\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}),$$

which reduces to Corollary 2.3 for $\alpha = 0$.

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